Topology of Spacetime

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ABSTRACT

Over very large length scales on the order of 300 million light years or so, the matter and energy that make up the universe take on a nearly-homogeneous isotopic distribution. As the observational scale decreases, a hierarchy of matter and energy distribution emerges first within galactic filaments, then superclusters, clusters, and so on moving down to atoms and elementary particles. On the largest of scales, the geometry of the local (observable) universe and beyond (global universe) remains yet unsolved. Observational evidence supports that global spacetime has zero curvature, is unbounded, and is infinite with a 0.4% margin of error. The experimental error with current data is large enough however that the sign on the curvature cannot be constrained, which only permits spacetime to be approximated as Euclidean.

1. Foundations for Defining Spacetime Topology

Within physical cosmology, it becomes very beneficial and convenient to work within a coordinate system known as comoving coordinates. At the most fundamental level, comoving coordinates are coordinates that factor out the expansion of the universe, such that a "comoving observer" would be moving with the Hubble flow. In doing so, this "comoving observer" would perceive the cosmic microwave background as isotropic, thereby eliminating the possibility of observing any aggregate sections of the sky as redshifted or blueshifted. The observable universe visible to this observer is then given by

$$\eta = \int_0^t \frac{dt'}{a(t')} \tag{1}$$

Where η is the conformal time, which is a monotonically increasing variable of time. Here we are looking at the light that could have traveled from the Big Bang at $t = 0$ to current time t. Our comoving distance then is $d\eta = \frac{dx}{a} = c\frac{dt}{a}$, with naturally defined $c = 1$, expansion rate a, and dx being a physically measurable distance. And so Equation 1 represents the sphere of the observable universe visible to the comoving observer, as shown in Figure 1. Having the necessary coordinate system defined, discussing the topology of spacetime is no longer such a mathematical chore. First, it is important to distinguish between the local geometry of spacetime and the global geometry of spacetime. The former concerns the topology of the observable universe, while the latter concerns the topology of the entire universe. So far it is unknown what exactly lies beyond the observable universe, but if the entire universe is encompassed by the observable universe it should be possible to determine the topology of the universe as a whole through observation. If however the observable universe is only one small part of the total universe, observations will reveal the topology of the observable universe and perhaps nearby regions of the total universe, but it would be fundamentally impossible to be able to draw any conclusions about how the entire universe is shaped. There is also the question of whether or not the universe is finite or infinite. A finite universe is defined where there exists a distance such that all possible points in space are within that distance of each other. By convention, the "diameter" of the universe is then defined as the smallest possible distance between which all points in space exist. It follows that a finite universe must then have a finite volume. In contrast, an infinite universe will have no definite volume. The universe is classified as infinite if there exists any two arbitrary points in space that can be found to be arbitrarily far apart. Knowing the difference between a finite and infinite universe permits the discussion of boundedness. The question of whether the universe is bounded or not - whether it has an edge or not - is partly dependent on whether or not the universe is finite. For the infinite universe case, it is trivial to conclude that there is then no edge to spacetime. However, a finite universe may or may not be bounded depending on its topology. For finite spaces such as a 3-sphere or 3-torus, the universe will be unbounded, but for something like a finite disc the universe will be bounded.

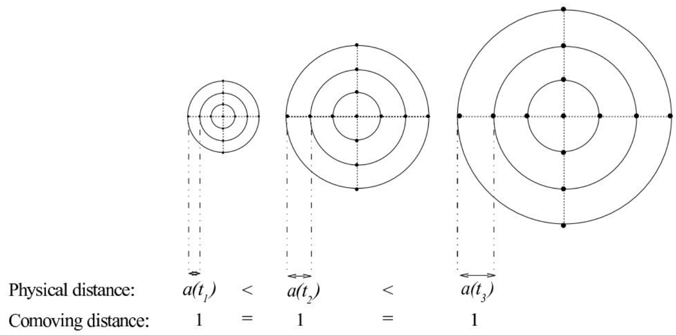

Fig. 1. Comoving and physical distances, with observer at the center of each circle. For the comoving case, the universe appears isotropic, but for the physical case the universe is noticably expanding.

2. Spacetime Curvature

Following from these foundations, the most important aspect that contributes to the understanding of the topology of spacetime is the curvature of the universe. There are three possible cases:

1. Spactime has zero curvature and is thus "flat" or Euclidean
2. Spacetime has positive curvature and is thus elliptic
3. Spacetime has negative curvature and is thus hyperbolic

Curvature is defined by the density of the universe. There is a common ratio called the "density parameter" that is employed when discussing curvature,

$$\Omega = \frac{\rho}{\rho_{crit}} \tag{2}$$

which comes from the relationship between the measured density of the universe ρ and the "critical density" ρ_{crit} which is the density required to have flat spacetime. This critical density is proportional to the square of the Hubble constant H_0 (note that the Hubble "constant" is not really a constant at all) at a given cosmological time and inversely proportional to Newton's gravitational constant G,

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} \approx 1.06 \cdot 10^{-26} \frac{kg}{m^3} \approx 6 \text{ hydrogen atoms per } m^3 \tag{3}$$

Both the density ρ and critical density ρ_{crit} can be applied to either the local or global topology of the universe, and so it necessarily must be mentioned either explicitly or made clear through context whether the densities that are being discussed are measurements of the local or global topologies. For $\Omega = 1$, the universe (local or global) is defined to be perfectly flat (see Figure 2), for $\Omega > 1$, the universe is defined to be positively curved (see Figure 3), and for $\Omega < 1$, the universe is defined to be negatively curved (see Figure 4).

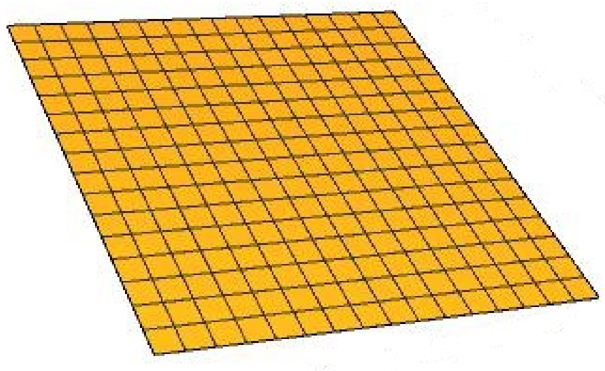

Fig. 2. General representation of "flat" or Euclidean spacetime. Here, spacetime has zero curvature evident in the fact that a triangle drawn on this plane will have angles that add up to 180°.

There are two methods through which the density of the universe can be determined:

1. Accounting approach: estimate the mass of a given volume of the universe
2. Geometrical approach: measure the convergence/divergence of parallel lines through large scale cosmic structures

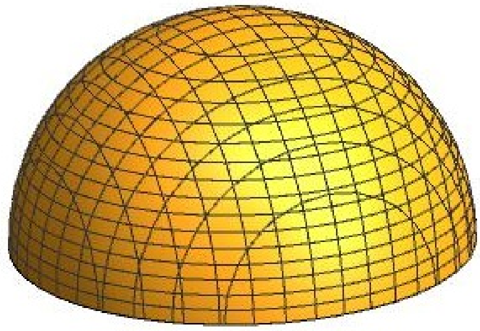

Fig. 3. General representation of elliptic spacetime. Here, spacetime has positive curvature. This is one of many possible positively-curved topological solutions that the universe (local or global) may take on.

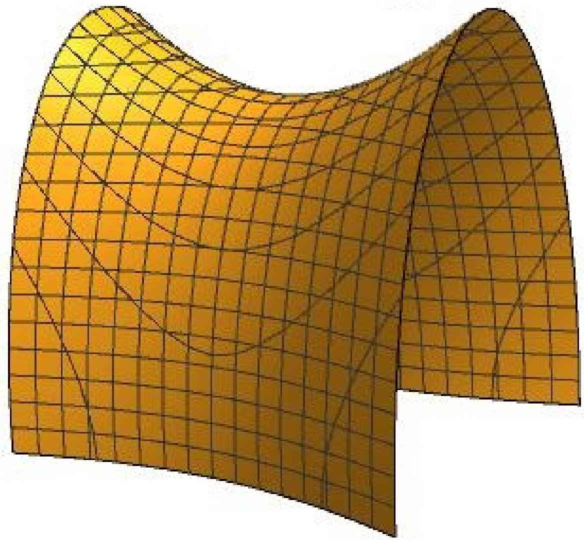

Fig. 4. General representation of hyperbolic spacetime. Here, spacetime has negative curvature. Like Figure 3, this is one of many possible topologies that the universe may take on and is certainly non-exhaustive of other possible solutions.

The accounting approach is the more brute-force method, and works by measuring the mass of objects within a specified volume of the universe. Mass measurements can be made either directly through the known kinematic properties of large cosmic bodies (e.g. galactic motions within clusters) or indirectly via galactic mass-luminosity relationships. The downfall to indirect measurement is that the method cannot account for dark matter, however educated assumptions on dark matter densities can be made via other avenues within current dark matter research. The geometrical approach takes a slightly more elegant approach and examines the path that light takes through very large scale cosmic structures. It works on the idea that for flat spacetime, parallel light rays will never diverge, while positively or negatively curved spacetime will see light rays that respectively converge or diverge. An alternate method that works on the same principle is to take a single light ray moving through a large scale structure, determine its propagation velocity between two points, and then take those two points in combination with Earth to form a very large triangle. If the angles of this triangle sum up to 180°, then it can be concluded that that portion of the universe has zero curvature as the Pythagorean theorem still applies. For positively curved spacetimes, the sum of the angles of said triangle will exceed 180°, and likewise negatively curved spacetimes will see that the three angles of said triangle do not sum all the way up to 180°. Current observations and research show that the universe is very nearly flat, which begs the question: why? There is no where in the standard model that says the universe must necessarily be flat or of any other curvature, so where does this flatness come about? This question is made even more puzzling if $\Omega = 1$, which tells us that the universe formed with perfectly flat curvature, as if conveniently balanced on a knife's edge between at least *some* curvature. This means that if the Universe started with $\Omega = 1$ exactly, it would remain so forever as there is no currently known mechanism for altering the large-scale curvature of the universe (perhaps the "micro"-perturbations of gravity within general relativity could shift the balance over very large time scales?). If however the universe was created with any other value of Ω, the equality in the density parameter would grow even more unequal as the Hubble constant changed through cosmological time, (also assuming the scale factor a grows slower than linearly in time). Following from the first Friedmann equation (derived from the 00 component of Einstein's field equations),

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \tag{4}$$

which can be rearranged and divided by critical mass (from critical density),

$$\rho_{crit} = \frac{m_{crit}}{\text{volume}} = \frac{3H^2}{8\pi G} \tag{5}$$

to yield

$$\Omega - 1 = \frac{\rho - \rho_{crit}}{\rho_{crit}} = \frac{k}{\dot{a}^2} \tag{6}$$

with intrinsic curvature k constant for a particular solution, a being the scale factor discussed previously, and k/a^2 defined as the spatial curvature of spacetime for a given cosmological time. And so for $t \to 0, \dot{a} \to \infty$, and thus $\Omega - 1 \to 0$.

Let $a = a_0(\frac{t}{t_0})^p$ with p an artifact to the solution of the differential equation to see that, t being a non-zero cosmological time, and t_0 being the reference time (to avoid issues with explicitly defining $t = 0$ for such an abstract event) at which time began (Big Bang):

$$\dot{a} = a_0 t_0^{-p} p^{p-1} \tag{7}$$

such that

$$\frac{k}{\dot{a}^2} = \frac{k}{a_0^2 p^2} t_0^{2p} p^2 t^{2(1-p)} =< k > t^{2(1-p)} \tag{8}$$

demonstrates that

$$\Omega - 1 =< k > t^{2(1-p)} \tag{9}$$

such that $\Omega - 1 \to 0$ as $t \to 0$ for $p < 1$.
and that $\Omega - 1 \to \infty$ as $t \to \infty$ for $p < 1$.

And so it is shown that the magnitude of Ω grows with increasing time t which suggests that over the entire history of the universe for which a scales less-than-linearly, the universe has been growing increasingly non-flat. This implies that the curvature of the universe is necessarily an unstable fixed point about $k = 1$ for $p < 1$ (that is, unless $\Omega = 1$ precisely).

3. Current Data and the FLRW Model

Current data from WMAP, SDSS, 2dF, COBE, Planck, and BOOMERanG and other telescopes suggests that the universe is isotropic and homogeneous as well as infinite and flat with a 0.4% margin of error. This new data is in agreement with older data (c. 2001) from experiments including MAT/TOCO, Maxima, DASI, and BOOMERanG, which found the universe to be flat with a margin of error around 15%. The theoretical model that best matches current data is the Friedman-Lemaitre-Robertson-Walker (FLRW) model, which treats matter within the universe as a fluid. This model is most accurate because it is great for approximating the local geometry of spacetime which is all we currently have data for. Ignoring dark energy, the curvature can be determined by measuring the average mass density of the universe assuming matter is evenly distributed. This assumption is justified by the large-scale universe being homogeneous and isotropic. The spacetime metric that is described by the FLRW model is given by

$$-c^2 d\tau^2 = -c^2 dt^2 + a(t)^2 d\Sigma^2 \tag{10}$$

with Σ defining the coordinate system that the FLRW metric is describing. For example, for Cartesian coordinates,
$d\Sigma^2 = dx^2 + dy^2 + dz^2$,
or for polar coordinates,
$d\Sigma^2 = \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$. The FLRW model provides an analytical solution to Einstein's field equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{11}$$

with solutions given by

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho \tag{12}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G}{3} p \tag{13}$$

While a wonderful approximation and currently the most well-fit model to current observational data, it should be noted that the FLRW model is unable to predict fine structure within the universe, as objects like galaxies, stars, planets, etc. are much more dense than the average density of the universe. For the most part however, it is accepted that on the large scale where the universe and spacetime can be taken to be isotropic, homogeneous, infinite, and flat, the FLRW model describes reality with enough accuracy that further interpretations can be made from the model with confidence that the model is "correct enough."

References

[1967] Mihalas, D., Alder, B., Fernback, S., & Rotenberg, M., eds, in Methods in Computational Physics, Vol. 7, The Calculation of Model Stellar Atmospheres

[2014] Terzic, Balsa, Northern Illinois University, Center for Accelerator and Detector Development, Cosmic Distances.

[1999] Friedmann, Alexander, "Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes," Zeitschrift für Physik A, 21 (1): 326-332, doi:10.1007/BF01328280. English translation in "General Relativity and Gravitation," 1999 vol.31, 31-

[2017] Swinburne University of Technology, Department of Astronomy, Critical Density, http://astronomy.swin.edu.au/cosmos/C/Critical+Density

[2014] National Aeronautics and Space Administration, Will the Universe Expand Forever, https://map.gsfc.nasa.gov

[2004] Aurich, Ralf; Lustig, S.; Steiner, F.; Then, H. "Hyperbolic Universes with a Horned Topology and the CMB Anisotropy." Classical and Quantum Gravity. 21 (21): 4901-4926. arXiv:astro-ph/0403597. Bibcode:2004CQGra..21.4901A. doi:10.1088/0264-9381/21/21/010

[2003] Demianski, Marek; Sánchez, Norma; Parijskij, Yuri N. "Topology of the universe and the cosmic microwave background radiation". The Early Universe and the Cosmic Microwave Background: Theory and Observations. Proceedings of the NATO Advanced Study Institute. The early universe and the cosmic microwave background: theory and observations. Springer. 130: 161. Bibcode:2003eucm.book..159D. ISBN 1-4020-1800-2.

[2017] Imamura, James, World Views of the Universe, http://hendrix2.uoregon.edu/ imamura/123/lecture-5/lecture-5.html.

[1999] G. F. R. Ellis; H. van Elst. "Cosmological models (Cargèse lectures 1998)". In Marc Lachièze-Rey. Theoretical and Observational Cosmology. NATO Science Series C. 541. p. 22. arXiv:grqc/9812046. Bibcode:1999toc..conf....1E. ISBN 978-0792359463.

[2006] 3DXM Consortium, Psuedospherical Surfaces, http://virtualmathmuseum.org/Surface/gallery$_o.htmlPseudosphericalSurfaces$

[1968] Mostow, G. D., "Quasi-conformal mappings in n-space and the rigidity of the hyperbolic space forms", Publ. Math. IHES, 34: 53–104, doi:10.1007/bf02684590

[2009] Vardanyan, Mihran; Trotta, Roberto; Silk, Joseph (2009). "How flat can you get? A model comparison perspective on the curvature of the Universe". Monthly Notices of the Royal Astronomical Society. 397: 431. arXiv:0901.3354. Bibcode:2009MNRAS.397..431V. doi:10.1111/j.1365-2966.2009.14938.x

[2015] Planck Collaboration; Ade, P. A. R., et al. (2015). "Planck 2015 results. XIII. Cosmological parameters". arXiv:1502.01589 [astro-ph.CO].